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GROUPS OF HIGHER DIMENSIONAL SATELLITE KNOTS

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1. Introduction

The group of a knot is the fundamental group of the knot complement. A knot with a companion is called a satellite knot. For the classical knots, Schubert [21], [22] developed a theory of companionship of knots, which generalized the classical construction of composition of knots [20]. The higher dimensional satellite knots were first considered by Shinohara [25], who generalized the results on the Alexander polynomials and the signatures of classical satellite knots [23], [24] to the higher dimensional case. Recently, Yoshikawa [29] constructed an example of a non-fibered 2-knot which has a group with a nontrivial center using a satellite knot. In this paper, we investigate the groups of higher dimensional satellite knots.

In Section 2, the precise definition of a satellite knot is given, and the group of a satellite knot and its commutator subgroup are discussed. Section 3 contains an application of a satellite knot.

The deficiency of a 2-knot group is always ≤ 1. Knots with deficiency one were constructed by Artin [2] and knots with deficiency zero by Fox [5]. Levine [15] have recently constructed knots of arbitrary large negative deficiency, by the composition of many knots of deficiency zero. In Section 3, we construct a satellite knot of any negative deficiency whose group is prime [17]. Modulo the unknotting conjecture for 2-knots, this is an affirmative answer to Problem 1.49 of Kirby [13], which was posed by S. Lomonaco.

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The remainder of this section is devoted to definitions and notational conventions.

We shall work throughout in the PL category. The boundary and interior of a manifold M are denoted by ∂M and $\operatorname{int} M$. R^n, S^n and D^n denote, respectively, the Euclidean n-space, n-sphere and n-disk.

An *n-knot* K is a locally flat oriented submanifold of the oriented $(n+2)$-sphere S^{n+2} which is homeomorphic to S^n. Throughout the paper we assume that the orientation of S^{n+2} is fixed.

Two n-knots K_1 and K_2 are *equivalent* if there exists an orientation preserving

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homeomorphism f of S^{n+2} onto itself such that $f(K_1)=K_2$ and $f(K_1)$ is orientation preserving. An n-knot is *trivial* if it bounds an $(n+1)$-disk in S^{n+2}.

Let G and H be groups. If G and H are isomorphic, then we write $G \cong H$. $G * H$ denotes the free product of G and H. The commutator subgroup of G is denoted by $[G]$.

Z denotes the infinite cyclic group or the integers. Z_n denotes the cyclic group of order n; Z_0 means the infinite cyclic group Z. N denotes the natural numbers.

We write πK for the group of an n-knot K, i.e. $\pi K = \pi_1(S^{n+2}-K)$. $N(K)$ denotes a tubular neighborhood of K in S^{n+2}, which is homeomorphic to $S^n \times D^2$ [27, p. 268]. We call $x \in \pi K$ a *meridian of* K if x is represented by an oriented loop $p \times D^2$, $p \in S^n$, such that the intersection number of D^2 and K, regarding D^2 as a 2-chain and K as an n-cycle, is $+1$ in $H_0(S^{n+2}) \cong Z$.

Let K be an n-knot and $G=\pi K$. When we call G an n-knot group, we always consider the pair G and K, so we shall use the notation πK rather than G. Instead of this pair, we may consider the pair G and a meridian of K, see [17].

2. Satellite knots

Let J and K be n-knots and w and x be meridians of J and K, respectively. Any element of πK can be written in the form $x^p\beta$, where $\beta \in [\pi K]$, and can be represented by a simple closed curve l in $S^{n+2}-N(K)$ by Corollary 8.1.2 of [8]. Let V be a tubular neighborhood of l in $S^{n+2}-N(K)$. Since l bounds a 2-disk in S^{n+2} by Theorem 10.1 of [8], $S^{n+2}-\operatorname{int} V$ is homeomorphic to $S^n \times D^2$. Let $h: S^{n+2}-\operatorname{int} V \to N(J)$ be a homeomorphism. Then we obtain an n-knot $h(K)$. Note that $h(K)$ is not dependent on l, but on $x^p\beta$, because every simple closed curve representing $x^p\beta$ is contained in a unique ambient isotopy class by Theorem 10.1 of [8]. Therefore we denote $h(K)$ by $\Sigma(J; K, x^p\beta)$, and call it the *satellite knot of* $(K, x^p\beta)$ *about* J, or simply a *satellite knot*. We call J a *companion* of $\Sigma(J; K, x^p\beta)$. If either $x^p\beta = 1$ or J is trivial, then it is clear that $\Sigma(J; K, x^p\beta)=K$.

By the van Kampen Theorem, we have the following diagram of inclusion homomorphisms:

$$\begin{tikzcd}
 & \pi_1(\partial N(J)) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
\pi_1(S^{n+2}-\operatorname{int} N(J)) \arrow[dr] & & \pi_1(N(J)-h(K)) \arrow[dl] \\
 & \pi_1(S^{n+2}-h(K)) &
\end{tikzcd}$$

Clearly we have $\pi_1(\partial N(J)) = \langle w \mid \ \rangle \cong Z$, $\pi_1(S^{n+2}-\operatorname{int} N(J)) \cong \pi J$, $\pi_1(S^{n+2}-h(K)) = \pi\Sigma(J; K, x^p\beta)$ and $i_1(w)=w$. Furthermore, since an inclusion homomorphism

$$\pi_1((S^{n+2}-\operatorname{int} V)-K) \to \pi K$$

is an isomorphism, we have

$$\pi_1(N(J)-h(K))\cong\pi_1((S^{n+2}-\operatorname{int} V)-K)\cong\pi K,$$

and $i_2(w)=x^p\beta$.

Let q be the order of $x^p\beta$ in πK; if it is infinite, then let $q=0$. Note that if $p\neq 0$ then $q=0$, so $pq=0$. Let $\langle w^q\rangle$ be the normal closure of w^q in πJ and $\widetilde{\pi J}=\pi J/\langle w^q\rangle$. Then the order of w in $\widetilde{\pi J}$ is q, since $\widetilde{\pi J}/[\widetilde{\pi J}]=\langle w\mid w^q=1\rangle\cong Z_q$. Then the subgroup of $\widetilde{\pi J}$ generated by w, which we denote by H, and the subgroup of πK generated by $x^p\beta$ are both isomorphic to Z_q. Thus $\pi\Sigma(J;K,x^p\beta)$ is the free product of $\widetilde{\pi J}$ and πK with an amalgamated subgroup H [18, p. 207]:

$$\pi\Sigma(J;K,x^p\beta)=\widetilde{\pi J}*_H\pi K,$$

and so it contains subgroups isomorphic to $\widetilde{\pi J}$ and πK.

Now we write down a presentation of $\pi\Sigma(J;K,x^p\beta)$ in terms of presentations of $[\pi J]$, $[\pi K]$, πJ and πK. Let

$$[\pi J]=\langle A\mid T\rangle \quad\text{and}\quad [\pi K]=\langle B\mid U\rangle,$$

where

$$A=\{a_\mu\mid\mu\in N\},\qquad B=\{b_\nu\mid\nu\in N\},$$
$$T=\{t_i(a_\mu)\mid i\in N\},\quad U=\{u_j(b_\nu)\mid j\in N\};$$

$t_i(a_\mu)$ is a word in A and $u_j(b_\nu)$ a word in B. Since πJ is a split extension of $[\pi J]$ by the infinite cyclic group $\pi J/[\pi J]=\langle w\mid\ \rangle$, πJ has the following presentation [7, 15.4]:

$$\pi J=\langle w,A\mid R,T\rangle,$$

where

$$R=\{wa_iw^{-1}=r_i(a_\mu)\mid i\in N\};$$

$r_i(a_\mu)$ is a word in A. Similarly, we have

$$\pi K=\langle x,B\mid S,U\rangle,$$

where

$$S=\{xb_jx^{-1}=s_j(b_\nu)\mid j\in N\};$$

$s_j(b_\nu)$ is a word in B. Hence

$$\pi\Sigma(J;K,x^p\beta)=\langle w,x,A,B\mid R,S,T,U,w=x^p\beta\rangle.$$

Theorem 2.1. *If $p>0$, then*

$$[\pi\Sigma(J;K,x^p\beta)]=\mathop{*}_{k=0}^{p-1}x^k[\pi J]x^{-k}*[\pi K],$$

and if $p<0$, then

$$[\pi\Sigma(J;K,x^p\beta)]=\mathop{*}_{k=p+1}^{0}x^k[\pi J]x^{-k}*[\pi K].$$

Proof. Suppose that $p>0$. Let $\alpha_{\mu_k}=x^k a_\mu x^{-k}$ and

$$A_k=\{\alpha_{\mu_k} \mid \mu\in N\}.$$

Since

$$wa_i w^{-1}=(x^p\beta x^{-p})\{x(x^{p-1}a_i x^{-(p-1)})x^{-1}\}(x^p\beta^{-1}x^{-p})$$
$$=(x^p\beta x^{-p})(x\alpha_{i,p-1}x^{-1})(x^p\beta^{-1}x^{-p})$$

and

$$a_\mu=\alpha_{\mu_0},$$

we have

$$x\alpha_{i,p-1}x^{-1}=(x^p\beta^{-1}x^{-p})r_i(\alpha_{\mu_0})(x^p\beta x^{-p}).$$

Let

$$\tilde{R}=\{x\alpha_{ik}x^{-1}=\alpha_{i,k+1}, x\alpha_{i,p-1}x^{-1}=(x^p\beta^{-1}x^{-p})r_i(\alpha_{\mu_0})(x^p\beta x^{-p}) \mid i\in N, k=0,1,\dots,p-2\}.$$

Let

$$T_k=\{t_i(\alpha_{\mu_k}) \mid i\in N\},$$

where $t_i(\alpha_{\mu_k})=x^k t_i(a_\mu)x^{-k}$. Then we have

$$\pi\Sigma(J;K,x^p\beta)=\langle x, A_0,\dots,A_{p-1},B \mid \tilde{R},S,T_0,\dots,T_{p-1},U\rangle.$$

Thus we obtain

$$[\pi\Sigma(J;K,x^p\beta)]=\langle A_0,\dots,A_{p-1},B \mid T_0,\dots,T_{p-1},U\rangle$$
$$=\langle A_0 \mid T_0\rangle*\cdots*\langle A_{p-1} \mid T_{p-1}\rangle*\langle B \mid U\rangle.$$

Since $\langle A_k \mid T_k\rangle=x^k[\pi J]x^{-k}$, the result follows. For the case $p<0$, we can show in the same way. □

If L is a fibered n-knot, then $[\pi L]$ is finitely generated [19, p. 324]. Yoshikawa [29] proved:

Proposition 2.2. *If $\widetilde{\pi J}\not\cong Z_q$, $\pi K\not\cong Z$ and $\beta\neq 1$, then $[\pi\Sigma(J;K,\beta)]$ is not finitely generated.*

In contrast with this proposition, we show the following theorem, which is a generalization of Theorem 5 of [26] to higher dimensions.

Theorem 2.3. *If J and K are fibered knots with fibers E and F, respectively and $p\neq 0$, then $\Sigma(J;K,x^p\beta)$ is a fibered knot with fiber a connected sum of $|p|$ copies of E and F; $\#_p E\# F$.*

Proof. Let l be a simple closed curve in $S^{n+2}-N(K)$ representing $x^p\beta$. We may suppose that l meets F transversely in $|p|$ points. Then $(S^{n+2}-\operatorname{int} V)-K$ fiberes over S^1 with fiber $F-\operatorname{int} V$, where V is a tubular neighborhood of l in $S^{n+2}-N(K)$. Thus the fiber of $\Sigma(J;K,x^p\beta)$ can be constructed by piecing together $|p|$ copies of E with $F-\operatorname{int} V$. □

A satellite knot $\Sigma(J; K, x)$ is the *composition of* J *and* K and denoted by $J\#K$, see [27, pp. 342–343]. An n-knot L is *prime* if $L = J\#K$ implies that either J or K is a trivial n-knot. We call $\pi(J\#K)$ the *composition of* πJ *and* πK, and denote it by $\pi J\#\pi K$ [17, p. 591]. An n-knot group πL is *prime* if $\pi L = \pi J\#\pi K$ implies either πJ or πK is isomorphic to Z. For $n\geq 3$, an n-knot L is not always prime though πL is prime, because there exists a nontrivial n-knot L with $\pi L \cong Z$ [14], [19, p. 285]. For 2-knots, modulo the unknotting conjecture [27, p. 342], a 2-knot L is prime if πL is prime.

By Theorem 2.1, we have immediately the following proposition due to Maeda [17, p. 592].

Proposition 2.4. *Let* $\pi L = \pi J\#\pi K$, *and let* $\langle[\pi J]\rangle$ *and* $\langle[\pi K]\rangle$ *be the normal closures of* $[\pi J]$ *and* $[\pi K]$ *in* πL, *respectively. Then*

(1) $[\pi L] = [\pi J] * [\pi K]$;

(2) $\pi J = \pi L/\langle[\pi K]\rangle$ *and* $\pi K = \pi L/\langle[\pi J]\rangle$.

3. The deficiency of a 2-knot groups

The deficiency of a finite presentation

$$\langle x_1, \dots, x_p \mid r_1, \dots, r_q\rangle$$

of a group G is defined as $p-q$. The *deficiency of a group* G, denoted by $\operatorname{def} G$, is the maximum of the deficiencies of its finite presentations.

Kervaire [11], [12] showed that any knot group G has the following properties:

(i) G is finitely presented.

(ii) $G/[G] \cong Z$.

(iii) The normal closure of some single element is G.

(iv) $H_2(G; Z) = 0$.

Conversely, he showed that for any $n\geq 3$, any group G with properties (i)–(iv) is the group of some n-knot. For $n=2$, he proved that if G satisfies (i)–(iii) and the additional property:

(iv)′ $\operatorname{def} G = 1$,

then there exists a homotopy 4-sphere Σ^4 and a locally flat embedding $K: S^2 \to \Sigma^4$ such that $G = \pi_1(\Sigma^4 - K(S^2))$.

If G is a knot group, then $\operatorname{def} G \leq 1$ [4]. Every spun 2-knot [2], [1], [19, p. 85] has a group with deficiency one, since its group is isomorphic to a 1-knot group. The 2-knot of Example 12 in [5] has a group with deficiency zero [11, p. 106]. Levine [15] constructed 2-knots of deficiency <0; he showed that $\operatorname{def} \pi F_m = 1 - m$, where F_m is the composition of m copies of the 2-twist spun trefoil [30].

Remark. The 2-knot of Example 12 in [5] and the 2-twist spun trefoil are equivalent [16], [10].

Accordingly Lomonaco asked in [13, Problem 1.49] whether a prime 2-knot can have deficiency <0. If the unknotting conjecture for 2-knots is correct, then Theorem 3.2 below gives an affirmative answer of this problem.

Lemma 3.1. (Compare [28, p. 322].) *Let T be the* 3*-twist spun trefoil. Let*

$$P_1 = \langle x, a \mid xax = ax^2a,\ x^3 = (ax)^3 \rangle$$

and

$$P_2 = \langle x, a, b \mid xax^{-1} = b,\ xbx^{-1} = ba,\ a^2 = b^2 = (ab)^2 \rangle.$$

Then $\pi T \cong P_1 \cong P_2$, where x is a meridian, and

$$[\pi T] = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle,$$

which is the quaternion group of order 8.

Proof. By [30], we have

$$\pi T = \langle x, y \mid xyx = yxy,\ y = x^{-3}yx^3 \rangle,$$

where x and y are meridians. This presentation indicates that x^3 is in the center of πT. From the first relation, we have $yxy^{-1} = x^{-1}yx$, and so $yx^3y^{-1} = x^{-1}y^3x$. Thus we have $x^3 = y^3$. Therefore

$$\pi T = \langle x, y \mid xyx = yxy,\ x^3 = y^3 \rangle.$$

Letting $y = ax$, we obtain $\pi T \cong P_1$.

Let $b = xax^{-1}$. From the second relation of P_1, we have $a(xax^{-1})(x^2ax^{-2}) = 1$, and so $xbx^{-1} = x^2ax^{-2} = b^{-1}a^{-1}$. From the first relation of P_1, we have $b = a(b^{-1}a^{-1})$ and $b^{-1}a^{-1} = ba$. Hence we obtain $\pi T \cong P_2$. □

Remark. πT is isomorphic to the group of the roll-spun figure eight knot [6]. Also πT is isomorphic to the group of K_1 in [9]. The question of whether these three knots are equivalent is open.

Theorem 3.2. *Let T be the* 3*-twist spun trefoil and*

$$\pi T = \langle x, a \mid xax = ax^2a,\ x^3 = (ax)^3 \rangle.$$

Let T_m be the composition of m cpoies of the 3*-twist spun trefoil. Let $K_m = \Sigma(T_m; T, x^2a)$. Then*

(1) $\operatorname{def} \pi K_m = -m$;

(2) *πK_m is prime.*

Proof. (1) πT_m has a presentation

$$\langle w, c_1, \ldots, c_m \mid wc_iw = c_iw^2c_i,\ w^3 = (c_iw)^3,\ i = i, \ldots, m \rangle.$$

Then πK_m is presented by

$$G_1=\langle x, a, c_1, \dots, c_m \mid xax=ax^2a, x^3=(ax)^3, x^2ac_ix^2a=c_i(x^2a)^2c_i, (x^2a)^3=(c_ix^2a)^3, i=1,\dots,m\rangle.$$

Thus $\operatorname{def} \pi K_m \geq -m$.

Let $\tilde{X}_m$ be the infinite cyclic covering space of S^4-K_m. Then $H_*(\tilde{X}_m)$ has a Λ-module structure, where Λ is the polynomial ring $Z[t, t^{-1}]$ [19, Chapter 7]. In [15], $H_*(\tilde{X}_m)$ is denoted by $H_*(\pi K_m; \Lambda)$. From the above presentation of πK_m, the method of [19, Chapter 7] shows that

$$H_1(\tilde{X}_m)=A\oplus\underbrace{B\oplus\cdots\oplus B}_{m},$$

where $A=\Lambda/(2, t^2+t+1)$ and $B=\Lambda(2, t^4+t^2+1)$. If we regard A and B as Λ_2-modules, where $\Lambda_2=\Lambda/(2)$, then $A=\Lambda_2/(t^2+t+1)$ and $B=\Lambda_2/(t^4+t^2+1)$. Note that $t^4+t^2+1=(t^2+t+1)(t^2-t+1)$. By the structure theorem for modules over the principal ideal domain Λ_2, we can see that $H_1(\tilde{X}_m)$ cannot be generated as a Λ-module by fewer than $p+1$ elements. Applying the proof of Theorem II of [15], we can show that $\operatorname{def} \pi K_m \leq -m$, so the result holds.

(2) By Lemma 3.1, letting $b=xax^{-1}$ and $d_i=xc_ix^{-1}$, we have

$$\pi T=\langle x, a, b \mid xax^{-1}=b, xbx^{-1}=ba, a^2=b^2=(ab)^2\rangle$$

and

$$\pi T_m=\langle w, c_1, d_1, \dots, c_m, d_m \mid wc_iw^{-1}=d_i, wd_iw^{-1}=d_ic_i, c_i^2=d_i^2=(c_id_i)^2, i=1,\dots,m\rangle.$$

Thus by Theorem 2.1, we have

$$[\pi K_m]=[\pi T_m]*x[\pi T_m]x^{-1}*[\pi T],$$

where

$$[\pi T]=\langle a, b \mid a^2=b^2=(ab)^2\rangle$$

and

$$[\pi T_m]=\langle c_1, d_1, \dots, c_m, d_m \mid c_i^2=d_i^2=(c_id_i)^2, i=1,\dots,m\rangle.$$

Let $Q_0=[\pi T]$, $Q_{2i-1}=\langle c_i, d_i \mid c_i^2=d_i^2=(c_id_i)^2\rangle$ and $Q_{2i}=xQ_{2i-1}x^{-1}$. Then

$$[\pi K_m]=\mathop{*}_{j=0}^{2m} Q_j.$$

Here each Q_j is isomorphic to the quaternion group, so Q_j is indecomposable relative to free products [18, p. 195].

Now we prove the primeness of πK_m. Assume that πK_m is not prime; $\pi K_m=\pi J_1 \# \pi J_2$, where neither πJ_1 nor πJ_2 is infinite cyclic group. Then by Proposition 2.4, $[\pi K_m]=[\pi J_1]*[\pi J_2]$. Since Q_{2i-1} and Q_{2i} are conjugate in πK_m, we may suppose that

$$[\pi J_1] = \mathop{*}_{j=0}^{2l-2} Q_j \quad \text{and} \quad [\pi J_2] = \mathop{*}_{j=2l-1}^{2m} Q_j,$$

where $1 \le l \le m$. By Proposition 2.4, we have

$$\pi J_1 = \langle x, a, c_1, \ldots, c_{l-1} \mid xax = ax^2a, x^3 = (ax)^3, x^2ac_ix^2a = c_i(x^2a)^2c_i, (x^2a)^3 = (c_ix^2a)^3, i = 1, \ldots, l-1 \rangle$$

and

$$\pi J_2 = \langle x, c_l, \ldots, c_m \mid x^2c_jx^2 = c_jx^4c_j, x^6 = (c_jx^2)^3, j = l, \ldots, m \rangle$$

Thus $\pi K_m = \pi J_1 \# \pi J_2$ is presented by

$$\begin{aligned} G_2 = \langle x, a, c_1, \ldots, c_m \mid\ & xax = ax^2a, x^3 = (ax)^3, \\ & x^2ac_ix^2a = c_i(x^2a)^2c_i, (x^2a)^3 = (c_ix^2a)^3, \\ & x^2c_jx^2 = c_jx^4c_j, x^6 = (c_jx^2)^3, \\ & i = 1, \ldots, l-1, j = l, \ldots, m \rangle. \end{aligned}$$

Therefore G_1 and G_2 must be isomorphic under the mapping $x \to x$, $a \to a$, $c_i \to c_i$.

Adding to G_1 and G_2 the relations $x^3 = c_1 = \cdots = c_{m-1}$, and letting $c = c_m$, we obtain

$$\tilde{G}_1 = \langle x, a, c \mid xax = ax^2a, x^3 = (ax)^3 = 1, x^2acx^2a = c(x^2a)^2c, (x^2a)^3 = (cx^2a)^3 \rangle$$

and

$$\tilde{G}_2 = \langle x, a, c \mid xax = ax^2a, x^3 = (ax)^3 = 1, x^2cx^2 = cx^4c, x^6 = (cx^2)^3 \rangle,$$

respectively. These two groups must be isomorphic under the mapping $x \to x$, $a \to a$, $c \to c$.

First ,we consider $\tilde{G}_1$. Letting $y = ax$ and $z = x^2a$, and eliminating a, we obtain

$$\tilde{G}_1 = \langle x, y, z, c \mid xyx = yxy, x^3 = y^3 = 1, zcz = cz^2c, z^3 = (cz)^3, z = x^2yx^2 \rangle.$$

Let

$$B_1 = \langle x, y \mid xyx = yxy, x^3 = y^3 = 1 \rangle.$$

Then by Lemma 3.1, B_1 is a factor group of the group of the 3-twist spun trefoil, and is isomorphic to

$$\langle x, y \mid xyx = yxy, x^3 = 1 \rangle,$$

which is the binary tetrahedral group of order 24 [3, p. 134], and is also isomorphic to the special linear group SL(2,3), the group of all 2×2 matrices of integers (mod 3) of determinant 1, under the mapping $x \to \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $y \to \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, see [3, p. 98]. Then x^2yx^2 is mapped to $\left(\begin{smallmatrix} 2 & 0 \\ 2 & 2 \end{smallmatrix}\right)$, so it has order 6 in B_1. Let

$$A_1 = \langle z, c \mid zcz = cz^2c, z^3 = (cz)^3, z^6 = 1 \rangle.$$

Then $A_1/[A_1] = \langle z \mid z^6 = 1 \rangle \cong Z_6$. Thus $\tilde{G}_1$ is the free product of A_1 and B_1 with an amalgamated subgroup $H_1 = \langle z \mid z^6 = 1 \rangle \cong Z_6$:

$$\tilde{G}_1 = A_1 *_{H_1} B_1.$$

Let us denote the center of a group G by $C(G)$. By Corollary 4.5 of [18], $C(\tilde{G}_1)=H_1\cap C(A_1)\cap C(B_1)$. Since the center of SL(2, 3) is

$$\{(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}),(\begin{smallmatrix}2&0\\0&2\end{smallmatrix})\}\cong Z_2$$

and $(x^2yx^2)^3(=z^3)$ is mapped to $(\begin{smallmatrix}2&0\\0&2\end{smallmatrix})$, $C(\tilde{G}_1)\cong Z_2$.

Next, we consider $\tilde{G}_2$. We have

$$\tilde{G}_2=\langle x,a,c\mid xax=ax^2a, x^2cx^2=cxc, x^3=(ax)^3=(cx^2)^3=1\rangle.$$

Let

$$A_2=\langle x,c\mid x^2cx^2=cxc, x^3=(cx^2)^3=1\rangle,$$

$$B_2=\langle x,a\mid xax=ax^2a, x^3=(ax)^3=1\rangle.$$

Then it is easy to see that both A_2 and B_2 are isomorphic to the binary tetrahedral group. Hence $\tilde{G}_2$ is the free product of A_2 and B_2 with an amalgamated subgroup $H_2=\langle x\mid x^3=1\rangle\cong Z_3$:

$$\tilde{G}_2=A_2 *_{H_2} B_2.$$

Thus as before $C(\tilde{G}_2)=H_2\cap C(A_2)\cap C(B_2)$, which is a trivial group, because $H_2\cong Z_3$ and $C(A_2)\cong C(B_2)\cong Z_2$.

Hence $\tilde{G}_1$ and $\tilde{G}_2$ are not isomorphic, and this contradiction establishes the primeness of πK_m. □

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